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Quasiprobability Distribution Functions of Squeezed Pair Coherent States

Xiang-Guo Meng · Ji-Suo Wang · Bao-Long Liang

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Abstract Using the entangled state representation of Wigner operator and some formulae related to the two-variable Hermite polynomials, the Wigner function of the squeezed pair coherent state (SPCS) and its two marginal distributions are derived. Based on the entangled Husimi operator introduced by Fan et al. (Phys. Lett. A 358:203, 2006) and the Weyl ordering invariance under similar transformations, we also obtain the Husimi function of the SPCS and its marginal distribution functions. The comparison between the two quasibability functions shows that, for the same amount of information included in two functions, the solving process of the Husimi function is simpler than that of the Wigner function.

Keywords Squeezed pair coherent state · Entangled state representation · Wigner function · Entangled Husimi operator · Husimi function

1 Introduction

Quasiprobability distribution functions are extremely useful in studying the statistical properties of quantized light fields in the quasiprobability phase space because these functions are related to the density matrix which provides a complete statistical description of some physical systems. Among these distribution functions Wigner function [1, 2] and Husimi function [3, 4] have been most widespreadly used in various branches physics. For a state, if its Wigner function has a negative part in the phase space, then such a state is said to be nonclassical. Two marginal distributions of Wigner function lead to measuring probability density in coordinate space and momentum space. However, Wigner distribution function itself is not a probability distribution due to being both positive and negative. To overcome

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X.-G. Meng (✉) · J.-S. Wang · B.-L. Liang
Department of Physics, Liaocheng University, Liaocheng, Shandong Province, 252059, P.R. China
e-mail: xgmeng@lcu.edu.cn

X.-G. Meng
e-mail: mengxiangguo1978@sina.com

this inconvenience, the Husimi distribution function is introduced [5–8], which is defined in a manner that guarantees it to be nonnegative. Furthermore, the Husimi function, as a Gaussian broaden version of the Wigner function, has a much simpler structure, so it is more convenient to obtain the function and use it to interpret the corresponding quantum state.

In the Fock space pair coherent state (PCS) is defined as [9, 10]

$$|q,\beta\rangle = C_q \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{(n+q)!n!}} |n+q,n\rangle , \tag{1}$$

where β is a complex parameter and q is a positive number. In fact, the state $|q,\beta\rangle$ is the common eigenvector of annihilates photons operator a_1a_2 in pair and the two-mode number-difference operator Q, where $[Q, a_1a_2] = 0$ and $[a_i, a_j^\dagger] = \delta_{ij}$ $(i, j = 1, 2)$. The normalization constant $C_q = [(i|\beta|)^{-q} J_q(2i|\beta|)]^{-1/2}$, where $J_q(x)$ is the ordinary q-order Bessel function. If operating the neat expression of the two-mode squeezing operator $S_2(r) = \exp[r(a_1^\dagger a_2^\dagger - a_1a_2)]$ on the PCS, we obtain a new two-mode correlated state, i.e., squeezed pair coherent state (SPCS),

$$S_2(r)\,|q,\beta\rangle = \|q,\beta\rangle . \tag{2}$$

The SPCS involves entanglement and also exhibits strong nonclassical properties such as various squeezing, many photon antibunching and phase property. In experiment the SPCS can be generated by an ensemble of the two-level atoms in squeezed states passing through the cavity [11, 12] or based on a two-mode squeezed photon number matching process, which employs weak cross-Kerr media and on/off detection [13]. The object of this paper is to find a concise approach for deriving the Wigner function and the Husimi function characterizing the entanglement of the state $\|q,\beta\rangle$. To achieve this aim, it would be convenient to use the entangled state representations and the entangled Wigner operator [14], as well as the newly introduced form of the entangled Husimi operator as a pure two-mode squeezed coherent state density matrix [7].

2 Wigner Function of the SPCS

In order to obtain the Wigner function of the SPCS, we first recall the entangled state $|\eta\rangle$ representation and its important features. In [14] Fan has shown that the entangled state $|\eta\rangle$ simultaneously obeys the eigenvector equations

$$(a_1 - a_2^\dagger)\,|\eta\rangle = \eta\,|\eta\rangle , \qquad (a_1^\dagger - a_2)\,|\eta\rangle = \eta^*\,|\eta\rangle , \tag{3}$$

where the state $|\eta\rangle$ is defined as

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle , \quad \eta = \eta_1 + i\eta_2. \tag{4}$$

Using (3) we can prove that the state $|\eta\rangle$ is the common eigenstate of two particles' relative position $(X_1 - X_2)$ and the total momentum $(P_1 + P_2)$ in two-mode Fock space, i.e.,

$$(X_1 - X_2)\,|\eta\rangle = \sqrt{2}\eta_1\,|\eta\rangle , \qquad (P_1 + P_2)\,|\eta\rangle = \sqrt{2}\eta_2\,|\eta\rangle , \tag{5}$$

where $X_i=(a_i^\dagger+a_i)/\sqrt{2}$ and $P_i=(a_i-a_i^\dagger)/\sqrt{2}\mathrm{i}$, $(i=1,2)$. Noting the normal ordering form of the two-mode vacuum state projector $|00\rangle\langle 00|=:e^{-a_1^\dagger a_1-a_2^\dagger a_2}:$, where the symbol $::$ denotes normal ordering, and using the technique of integration within an ordered product (IWOP) of operator, we can immediately obtain the orthonormalized property and complete relation of the state $|\eta\rangle$

$$\langle\eta|\,\eta'\rangle=\pi\delta(\eta-\eta')\delta(\eta^*-\eta'^*),\qquad \int\frac{d^2\eta}{\pi}|\eta\rangle\langle\eta|=1. \tag{6}$$

Using the generating function formula of the two-variable Hermite polynomials $H_{m,n}(\chi,\chi^*)$ [14]

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}\left(\chi,\chi^*\right)=\exp\left(-tt'+t\chi+t'\chi^*\right), \tag{7}$$

where

$$\begin{aligned}H_{m,n}\left(\chi,\chi^*\right)&=\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\exp\left(-tt'+t\chi+t'\chi^*\right)\Big|_{t=t'=0}\\&=\sum_{l=0}^{\min(n,m)}\frac{(-)^l\,n!m!}{l!\,(m-l)!\,(n-l)!}\chi^{m-l}\chi^{*n-l},\end{aligned} \tag{8}$$

which yields

$$H^*_{m,n}\left(\chi,\chi^*\right)=H_{n,m}\left(\chi,\chi^*\right), \tag{9}$$

thus the entangled state $|\eta\rangle$ can be expanded as

$$|\eta\rangle=e^{-|\eta|^2/2}\sum_{m,n=0}^{\infty}\frac{(-1)^n}{\sqrt{m!n!}}H_{m,n}\left(\eta,\eta^*\right)|m,n\rangle, \tag{10}$$

where $|m,n\rangle=\frac{a_1^{\dagger m}a_2^{\dagger n}}{\sqrt{m!n!}}|00\rangle$ is the two-mode number state. In the state $|\eta\rangle$ representation, using the IWOP technique the two-mode entangled Wigner operator is

$$\Delta\left(\rho,\gamma\right)=\int\frac{d^2\eta}{\pi^3}|\rho-\eta\rangle\langle\rho+\eta|\exp\left(\eta\gamma^*-\eta^*\gamma\right), \tag{11}$$

where $\gamma=\epsilon-\varepsilon^*$, $\rho=\epsilon+\varepsilon^*$, $\epsilon=(x_1+ip_1)/\sqrt{2}$ and $\varepsilon=(x_2+ip_2)/\sqrt{2}$. And also the standard two-mode squeezing operator $S_2(r)$ can be changed into an integration projection operator, i.e.,

$$S_2(r)=\int\frac{d^2\eta}{\mu\pi}|\eta/\mu\rangle\langle\eta|,\quad \mu=e^r. \tag{12}$$

If operating $S_2(r)$ on the state $|\eta\rangle$ we have

$$S_2\left(r\right)|\eta\rangle=\frac{1}{\mu}|\eta/\mu\rangle. \tag{13}$$

So using (11)–(13) the entangled Wigner operator becomes

$$S_2^{-1}(r)\,\Delta(\rho,\gamma)\,S_2(r)=\mu^2\int\frac{d^2\eta}{\pi^3}\,|\mu(\rho-\eta)\rangle\langle\mu(\rho+\eta)|\exp\left(\eta\gamma^*-\eta^*\gamma\right). \tag{14}$$

Making use of (2), (9), (10) and (13), we derive the overlap between $\langle\eta|$ and $\|q,\beta\rangle$,

$$\langle\eta\,\|q,\beta\rangle=\mu\langle\mu\eta\,|q,\beta\rangle=\mu C_q e^{-|\mu\eta|^2/2}\sum_{n=0}^{\infty}H_{n,q+n}\left(\mu\eta,\mu\eta^*\right)\frac{(-\beta)^n}{n!\,(q+n)!}. \tag{15}$$

If choosing $\eta=|\eta|e^{i\varphi}$, we have

$$\langle\eta\,\|q,\beta\rangle=\mu C_q e^{-iq\varphi}e^{-|\mu\eta|^2/2}\sum_{n=0}^{\infty}H_{n,q+n}\left(\mu|\eta|,\mu|\eta^*|\right)\frac{(-\beta)^n}{n!\,(q+n)!}. \tag{16}$$

Using (1), (11) and (15), we get the Wigner function of the state $\|q,\beta\rangle$

$$\begin{aligned}
W(\rho,\gamma)&=\langle q,\beta\|\,\Delta(\rho,\gamma)\,\|q,\beta\rangle\\
&=\mu^2C_{q,n}C_{q,m}^*\int\frac{d^2\eta}{\pi^3}\langle q,\beta\,|\mu(\rho-\eta)\rangle\langle\mu(\rho+\eta)|\,q,\beta\rangle\exp\left(\eta\gamma^*-\eta^*\gamma\right)\\
&=\mu^2C_{q,n}C_{q,m}^*\sum_{m,n=0}^{\infty}\frac{(-)^{n+m}\beta^{*m}\beta^n}{m!n!\,(q+n)!\,(q+m)!}\\
&\quad\times\int\frac{d^2\eta}{\pi^3}H_{q+m,m}[\mu(\rho-\eta),\mu(\rho-\eta)^*]H_{n,q+n}[\mu(\rho+\eta),\mu(\rho+\eta)^*]\\
&\quad\times\exp(-|\mu\gamma|^2-|\mu\eta|^2+\eta\gamma^*-\eta^*\gamma)\\
&=\mu^2C_{q,n}C_{q,m}^*\sum_{m,n=0}^{\infty}\frac{(-)^{n+m}\beta^{*m}\beta^n}{m!n!\,(q+n)!\,(q+m)!}\frac{\partial^{q+2m}}{\partial t^{q+m}\partial t'^m}\frac{\partial^{q+2n}}{\partial r^n\partial r'^{q+n}}\\
&\quad\times\int\frac{d^2\eta}{\pi^3}\exp[-|\mu\gamma|^2-|\mu\eta|^2+\eta\gamma^*-\eta^*\gamma-tt'+t\mu(\rho-\eta)\\
&\quad+t'\mu(\rho-\eta)^*-rr'+r\mu(\rho+\eta)+r'\mu(\rho+\eta)^*]|_{t=t'=r=r'=0}.
\end{aligned} \tag{17}$$

Further, using the following integration formula [14]

$$\int\frac{d^2z}{\pi}\exp(\zeta\,|z|^2+\xi z+\eta z^*)=\frac{1}{\zeta}\exp\left(\frac{-\xi\eta}{\zeta}\right),\qquad \operatorname{Re}\zeta<0 \tag{18}$$

and (8), the relatively compact form of the Wigner function is obtained

$$\begin{aligned}
W(\rho,\gamma)=&\sum_{m,n=0}^{\infty}\frac{C_{q,n}C_{q,m}^*(-)^{n+m}\beta^{*m}\beta^n}{\pi^2m!n!\,(q+n)!\,(q+m)!}\exp(-\mu^2|\rho|^2-|\gamma|^2/\mu^2)\\
&\times H_{q+m,q+n}[(\mu\rho+\gamma/\mu),(\mu\rho+\gamma/\mu)^*]H_{n,m}[(\mu\rho-\gamma/\mu),(\mu\rho-\gamma/\mu)^*].
\end{aligned} \tag{19}$$

Through further calculation, we find that the Wigner function $W(\rho,\gamma)$ of the state $\|q,\beta\rangle$ can be written as a sum of the mixture part $W^M(\rho,\gamma)$ and the quantum interference part $W^I(\rho,\gamma)$:

$$W(\rho,\gamma)=W^M(\rho,\gamma)+W^I(\rho,\gamma), \tag{20}$$

where the mixture part $W^M(\rho,\gamma)$ has the following form

$$\begin{aligned}W^M(\rho,\gamma)=&\frac{1}{\pi^2}\left|C_{q,m}\right|^2\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{l=0}^{q+m}\frac{|\beta|^{2m}(-1)^{l+k}}{l!k![(m-k)!(q+m-l)!]^2}\\&\times|\mu\rho-\gamma/\mu|^{2(m-k)}|\mu\rho+\gamma/\mu|^{2(q+m-l)}\exp(-\mu^2|\rho|^2-|\gamma|^2/\mu^2)\end{aligned} \tag{21}$$

and the interference part $W^I(\rho,\gamma)$ is

$$\begin{aligned}W^I(\rho,\gamma)=&\frac{1}{\pi^2}C_{q,n}C_{q,m}^*\left(\sum_{n>m}\sum_{k=0}^{m}\sum_{l=0}^{q+m}+\sum_{m>n}\sum_{k=0}^{n}\sum_{l=0}^{q+n}\right)\\&\times\frac{(-1)^{n+m+l+k}\beta^{*m}\beta^n}{k!l!(m-k)!(n-k)!(q+n-l)!(q+m-l)!}(\mu\rho-\gamma/\mu)^{*m-k}\\&\times(\mu\rho-\gamma/\mu)^{n-k}(\mu\rho+\gamma/\mu)^{q+m-l}(\mu\rho+\gamma/\mu)^{*q+n-l}\\&\times\exp(-\mu^2|\rho|^2-|\gamma|^2/\mu^2).\end{aligned} \tag{22}$$

The quantum interference part $W^I(\rho,\gamma)$ gives rise to the phase sensitive nonclassical effects, such as quadrature squeezing, which are associated with the variances of the quadrature fluctuations. By reason that μ plays the role of a squeezing parameter, when $\mu>1$, we clearly see that the Wigner distribution can be compressed along the γ direction at the expanse of an increase along the ρ direction. However, when $\mu<1$, the compressed trend is just contrary to the conclusions for $\mu>1$. Moveover, the comparison (21) and (22) show that the quantum interference effects only rely on the interference part $W^I(\rho,\gamma)$. In conclusion, the behaviour of Wigner function $W(\rho,\gamma)$ is in a good agreement with the quantum features of the SPCS.

3 Marginal Distributions of the Wigner Function of $\|q,\beta\rangle$

In another entangled state $|\xi\rangle$ representation, the two-mode Wigner operator is [14]

$$\Delta(\rho,\gamma)=\int\frac{d^2\xi}{\pi^3}|\rho-\xi\rangle\langle\rho+\xi|\exp\left(\xi^*\gamma-\xi\gamma^*\right), \tag{23}$$

where the explicit expression of the state $|\xi\rangle$ is

$$|\xi\rangle=\exp\left[-\frac{1}{2}|\xi|^2+\xi a_1^\dagger+\xi^*a_2^\dagger-a_2^\dagger a_1^\dagger\right]|00\rangle,\quad \xi=\xi_1+i\xi_2, \tag{24}$$

which is the common eigenstate of two particles' center-of-mass coordinate (X_1+X_2) and the relative momentum (P_1-P_2) in two-mode Fock space. Using the IWOP technique we easily get the orthonormalized and complete property of the state $|\xi\rangle$

$$\langle\xi|\xi'\rangle=\pi\delta(\xi-\xi')\delta(\xi^*-\xi'^*),\qquad\int\frac{d^2\xi}{\pi}|\xi\rangle\langle\xi|=1. \tag{25}$$

Performing the integration of $\Delta(\rho,\gamma)$ in (23) over $d^2\gamma$ leads to a projection operator, i.e.,

$$\int d^2\gamma\,\Delta(\rho,\gamma) = \frac{1}{\pi}\,|\xi\rangle\,\langle\xi|_{\xi=\rho}\,, \tag{26}$$

so $\Delta(\rho,\gamma)$ is named as the entangled Wigner operator. Thus we obtain a marginal distribution of the Winger function of the state $\|q,\beta\rangle$ in the ρ variable

$$\begin{aligned}\int d^2\gamma\,W(\rho,\gamma) &= \frac{1}{\pi}\,|\langle\xi\ \|q,\beta\rangle|^2_{\xi=\rho}\\ &= \frac{C_q^2 e^{-|\rho|^2/\mu^2}}{\mu^2\pi}\left|\sum_{n=0}^{\infty} H_{n,q+n}\left(\rho/\mu,\rho^*/\mu\right)\frac{\beta^n}{n!\,(q+n)!}\right|^2,\end{aligned} \tag{27}$$

where we have used the expansion of the entangled state $|\xi\rangle$ in the Fock space

$$|\xi\rangle = e^{-|\xi|^2/2}\sum_{m,n=0}^{\infty}\frac{1}{\sqrt{m!n!}}H_{m,n}\left(\xi,\xi^*\right)|m,n\rangle \tag{28}$$

and the relation between $S_2(r)$ and $|\xi\rangle$

$$S_2(r)\,|\xi\rangle = \mu\int\frac{d^2\xi'}{\pi}|\mu\xi'\rangle\langle\xi'|\xi\rangle = \mu|\mu\xi\rangle. \tag{29}$$

From (23) carrying out the integral over $d^2\rho$ for $\Delta(\rho,\gamma)$ yields

$$\int d^2\rho\,\Delta(\rho,\gamma) = \frac{1}{\pi}\,|\eta\rangle\,\langle\eta|_{\eta=\gamma}\,. \tag{30}$$

Similarly, using (15) we have

$$\begin{aligned}\int d^2\rho\,W(\rho,\gamma) &= \frac{1}{\pi}\,|\langle\eta\ \|q,\beta\rangle|^2_{\eta=\gamma}\\ &= \frac{\mu^2 C_q^2}{\pi}e^{-\mu^2|\gamma|^2}\left|\sum_{n=0}^{\infty} H_{n,q+n}\left(\mu\gamma,\mu\gamma^*\right)\frac{(-\beta)^n}{n!\,(q+n)!}\right|^2,\end{aligned} \tag{31}$$

which is another marginal distribution of the Wigner function of the SPCS in the γ variable. Equation (27) (or (31)) is proportional to the probability for finding the two particles, which have centre-of-mass position $\sqrt{2}\rho_1$(or relative position $\sqrt{2}\gamma_1$) and simultaneously relative momentum $\sqrt{2}\rho_2$ (or total momentum $\sqrt{2}\gamma_2$), under the SPCS.

4 Husimi Function of the SPCS

As a important extension of K. Husimi's work [5], Fan et al. [7] have constructed a new kind of entangled Husimi operator in order to exhibit the entanglement of some quantum states using the corresponding Husimi functions. Here the SPCS is just the highly correlated nonclassical state, so we use the entangled Husimi operator $\Delta_h(\rho,\gamma;\kappa)$ to obtain its

Husimi function. In [7] the operator $\Delta_h(\rho,\gamma;\kappa)$ is constructed based on the new two-mode squeezed coherent state expressed as

$$\begin{aligned}|\rho,\gamma;\kappa\rangle &= S_2^{-1}(\ln 1/\sqrt{\kappa})D_1(\alpha_1)D_2(\alpha_2)|0\rangle \\ &= \frac{2\sqrt{\kappa}}{1+\kappa}\exp\left\{\frac{-1}{1+\kappa}\left[\frac{\kappa}{2}|\rho|^2+\frac{|\gamma|^2}{2}-(\kappa\rho+\gamma)a_1^\dagger \right.\right.\\ &\quad\left.\left. +(\kappa\rho^*-\gamma^*)a_2^\dagger+(1-\kappa)\,a_1^\dagger a_2^\dagger\right]\right\}|00\rangle\,, \end{aligned} \tag{32}$$

where the consequential phase factor $i\frac{1-\kappa}{4(1+\kappa)}(\rho\gamma^*-\gamma\rho^*)$ is neglected, $S_2(\ln 1/\sqrt{\kappa})$ is the standard two-mode squeezed operator and $D_i(z_i)=\exp(\alpha_i a_i^\dagger-\alpha_i^* a_i)$ is the displaced operator, $\alpha_1=\frac{1}{2}(\gamma/\sqrt{\kappa}+\sqrt{\kappa}\rho)$, $\alpha_2=\frac{1}{2}(\gamma^*/\sqrt{\kappa}-\sqrt{\kappa}\rho^*)$, the positive parameter κ is the damping factor.

By virtue of the definition of the entangled two-mode Husimi operator which is smoothing out the entangled Wigner operator $\Delta_w(\rho',\gamma')$ by averaging over a "coarse graining" function

$$\Delta_h(\rho,\gamma;\kappa)=4\int d^2\rho' d^2\gamma'\,\Delta_w(\rho',\gamma')\exp\left[-\kappa|\rho'-\rho|^2-\frac{|\gamma'-\gamma|^2}{\kappa}\right], \tag{33}$$

where $\Delta_w(\rho',\gamma')$ is the two-mode entangled Wigner operator,

$$\begin{aligned}\Delta_w(\rho',\gamma') &= \frac{1}{\pi^2}:\exp\left[-(\rho'-a_1+a_2^\dagger)(\rho'^*-a_1^\dagger+a_2)\right.\\ &\quad\left.-\left(a_1+a_2^\dagger-\gamma'\right)\left(a_1^\dagger+a_2-\gamma'^*\right)\right]:, \end{aligned} \tag{34}$$

one see that the entangled Husimi operator is just defined as

$$\begin{aligned}\Delta_h(\rho,\gamma;\kappa) &\equiv |\rho,\gamma;\kappa\rangle\,\langle\rho,\gamma;\kappa| \\ &= \frac{4\kappa}{(1+\kappa)^2}:\exp\left[\frac{-1}{1+\kappa}\left(a_1+a_2^\dagger-\gamma\right)\left(a_1^\dagger+a_2-\gamma^*\right)\right.\\ &\quad\left.-\frac{\kappa}{1+\kappa}\left(\rho-a_1+a_2^\dagger\right)\left(\rho^*-a_1^\dagger+a_2\right)\right]:. \end{aligned} \tag{35}$$

Due to the Weyl ordered form of the entangled Husimi operator

$$\begin{aligned}\Delta_h(\rho,\gamma;\kappa) &= 4\,\vdots\exp\left[\kappa\left(a_1-a_2^\dagger-\rho\right)\left(a_1^\dagger-a_2-\rho^*\right)\right.\\ &\quad\left.-\frac{1}{\kappa}\left(a_1+a_2^\dagger-\gamma\right)\left(a_1^\dagger+a_2-\gamma^*\right)\right]\vdots, \end{aligned} \tag{36}$$

where the symbol $\vdots\ \vdots$ denotes Weyl ordering, and the squeezed operator $S_2(r)$ causes the following transformation

$$S_2^{-1}(Q_1+Q_2)\,S_2=(Q_1+Q_2)/\mu,\qquad S_2^{-1}(P_1+P_2)\,S_2=\mu\,(P_1+P_2)\,, \tag{37}$$

$$S_2^{-1}(Q_1-Q_2)\,S_2=\mu\,(Q_1-Q_2)\,,\qquad S_2^{-1}(P_1-P_2)\,S_2=(P_1-P_2)/\mu, \tag{38}$$

as well as the Weyl ordering invariance under similar transformations, we obtain

$$S_2^{-1}(r)\Delta_h(\rho,\gamma;k)S_2(r)=\Delta_h(\rho/u,u\gamma;\kappa u^2). \tag{39}$$

Now we first calculate the following inner product

$$\begin{aligned}
&\langle q,\beta|\rho/u,u\gamma;\kappa u^2\rangle\\
&=C_q^*\frac{2u\sqrt{\kappa}}{1+u^2\kappa}\sum_{n=0}^{\infty}\frac{\beta^{*n}}{(n+q)!n!}\langle 00|\,z_1^{n+q}z_2^n\int\frac{d^2z_1d^2z_2}{\pi^2}\,|z_1z_2\rangle\,\langle z_1z_2|\exp\left\{\frac{-1}{1+u^2\kappa}\right.\\
&\quad\times\left.\left[\frac{\kappa}{2}|\rho|^2+\frac{u^2|\gamma|^2}{2}-u(\kappa\rho+\gamma)z_1^*+u\left(\kappa\rho^*-\gamma^*\right)z_2^*+\left(1-u^2\kappa\right)z_1^*z_2^*\right]\right\}|00\rangle\\
&=C_q^*\frac{2u\sqrt{\kappa}}{1+u^2\kappa}\sum_{n=0}^{\infty}\frac{\beta^{*n}}{(n+q)!n!}\exp\left[\frac{-1}{1+u^2\kappa}\left(\frac{\kappa}{2}|\rho|^2+\frac{u^2|\gamma|^2}{2}\right)\right]\\
&\quad\times\int\frac{d^2z_1}{\pi}z_1^{n+q}\exp\left[-|z_1|^2+\frac{\kappa\rho+\gamma}{1+u^2\kappa}uz_1^*-\frac{1-u^2\kappa}{1+u^2\kappa}z_1^*z_2^*\right]\\
&\quad\times\int\frac{d^2z_2}{\pi}z_2^n\exp\left(-|z_2|^2-\frac{\kappa\rho^*-\gamma^*}{1+u^2\kappa}uz_2^*\right).
\end{aligned}\tag{40}$$

In (40) letting the following part as H, i.e.,

$$\begin{aligned}
H&=\int\frac{d^2z_1}{\pi}z_1^{n+q}\exp\left[-|z_1|^2+\frac{\kappa\rho+\gamma}{1+u^2\kappa}uz_1^*-\frac{1-u^2\kappa}{1+u^2\kappa}z_1^*z_2^*\right]\\
&=\frac{\partial^{n+q}}{\partial\chi^{n+q}}\int\frac{d^2z_1}{\pi}\exp\left[-|z_1|^2+\chi z_1+\frac{u(\kappa\rho+\gamma)+(u^2\kappa-1)z_2^*}{1+u^2\kappa}z_1^*\right]_{\chi=0}\\
&=\frac{\partial^{n+q}}{\partial\chi^{n+q}}\exp\left[\chi\frac{u(\kappa\rho+\gamma)+(u^2\kappa-1)z_2^*}{1+u^2\kappa}\right]_{\chi=0}\\
&=\left[\frac{u(\kappa\rho+\gamma)}{1+u^2\kappa}+\frac{u^2\kappa-1}{1+u^2\kappa}z_2^*\right]^{n+q}
\end{aligned}\tag{41}$$

and letting another part as Q, i.e.,

$$\begin{aligned}
Q&=\int\frac{d^2z_2}{\pi}z_2^n\exp\left(-|z_2|^2-\frac{\kappa\rho^*-\gamma^*}{1+u^2\kappa}uz_2^*\right)*H\\
&=\sum_{k=0}^{n+q}\binom{n+q}{k}\binom{n}{k}k!\frac{u^{2n+q-2k}(\kappa\rho+\gamma)^{n+q-k}\left(u^2\kappa-1\right)^k(\gamma^*-\kappa\rho^*)^{n-k}}{\left(1+u^2\kappa\right)^{2n+q-k}},
\end{aligned}\tag{42}$$

where we have used the overcomplete relation of two-mode coherent state $|z_1z_2\rangle$ [15] and the following integral formula [16]

$$\int\frac{d^2z}{\pi}z^{*n}z^m\exp\left(\zeta|z|^2+\eta z\right)=(-1)^{n+1}\zeta^{-(n+1)}\frac{n!\eta^{n-m}}{(n-m)!},\quad\text{Re}\,\zeta<0,m\le n. \tag{43}$$

Therefore using (35) the Husimi function for $\|q,\alpha\rangle$ can be immediately derived

$$\begin{aligned}
&|\langle q,\beta\|\,\rho,\gamma;\kappa\rangle|^2\\
&= \frac{4u^2\kappa\left|C_q\right|^2}{(1+u^2\kappa)^2}\exp\left[\frac{-(\kappa|\rho|^2+u^2|\gamma|^2)}{1+u^2\kappa}\right]\left|\sum_{n=0}^{\infty}\sum_{k=0}^{n+q}\binom{n}{k}\frac{\beta^{*n}}{n!(n+q-k)!}\right.\\
&\quad\times\left.\frac{u^{2n+q-2k}(\gamma+\kappa\rho)^{n+q-k}(u^2\kappa-1)^k(\gamma^*-\kappa\rho^*)^{n-k}}{(1+u^2\kappa)^{2n+q-k}}\right|^2,
\end{aligned} \tag{44}$$

which shows that the variations of the Husimi distribution is similar to that of the Wigner function when the squeezing parameter μ varies. Because the Husimi distribution is defined in a manner that it is smoothing out the entangled Wigner distribution by averaging over a "coarse graining" function, the information along the γ direction is weaken while the information along the ρ direction is enhanced for the enough big values of the damping factor κ.

5 Marginal Distributions of Husimi Function of $\|q,\beta\rangle$

In the present section we continue to derive the marginal distributions of the entangled Husimi function of the state $\|q,\beta\rangle$. Performing the integration of $\Delta_h(\rho,\gamma;\kappa)$ over $d^2\gamma$ leads to the entangled operator, i.e.,

$$\int\frac{d^2\gamma}{4\pi}\Delta_h(\rho,\gamma;\kappa)=\kappa e^{-\kappa[(\rho_1-\frac{X_1-X_2}{\sqrt{2}})^2+(\rho_2-\frac{P_1+P_2}{\sqrt{2}})^2]}, \tag{45}$$

then the marginal distribution of entangled Husimi function of the state $\|q,\beta\rangle$ in "ρ-direction" is

$$\begin{aligned}
P(\rho) &= \int\frac{d^2\gamma}{4\pi}\langle q,\beta\|\,\Delta_h(\rho,\gamma;\kappa)\,\|q,\beta\rangle\\
&= \langle q,\beta\|\,\kappa e^{-\kappa[(\rho_1-\frac{X_1-X_2}{\sqrt{2}})^2+(\rho_2-\frac{P_1+P_2}{\sqrt{2}})^2]}\|q,\beta\rangle\\
&= \langle q,\beta\|\int\frac{d^2\eta}{\pi}\kappa e^{-\kappa[(\rho_1-\eta_1)^2+(\rho_2-\eta_2)^2]}|\eta\rangle\langle\eta\|q,\beta\rangle,
\end{aligned} \tag{46}$$

where we have employed the completeness in (6). Now using (15) we obtain

$$\begin{aligned}
P(\rho) &= \kappa\mu^2C_q^2\sum_{m,n=0}^{\infty}\frac{(-\beta)^n(-\beta^*)^m}{n!\,(q+n)!m!\,(q+m)!}\\
&\quad\times\int\frac{d^2\eta}{\pi}H_{n,q+n}\left(\mu\eta,\mu\eta^*\right)H_{q+m,m}\left(\mu\eta,\mu\eta^*\right)\\
&\quad\times\exp[-\mu^2|\eta|^2-\kappa|\rho-\eta|^2],
\end{aligned} \tag{47}$$

where the integration is a fundamental one but will finally lead to a complicated expression, so we do not list it here. Similarly, performing the integration of $\Delta_h(\rho,\gamma;\kappa)$ over $d^2\rho$ leads to the entangled operator, i.e.,

$$\int \frac{d^2\rho}{4\pi}\Delta_h(\rho,\gamma;\kappa) = \frac{1}{\kappa}e^{-\frac{1}{\kappa}[(\gamma_1-\frac{X_1+X_2}{\sqrt{2}})^2+(\gamma_2-\frac{P_1-P_2}{\sqrt{2}})^2]}, \tag{48}$$

using (25) we also derive another marginal distribution of the entangled Husimi function of $\|q,\beta\rangle$ in "γ-direction"

$$\begin{aligned} P(\gamma) &= \int \frac{d^2\rho}{4\pi}\langle q,\beta\|\,\Delta_h(\rho,\gamma;\kappa)\,\|q,\beta\rangle \\ &= \langle q,\beta\|\,\frac{1}{\kappa}e^{-\frac{1}{\kappa}[(\gamma_1-\frac{X_1+X_2}{\sqrt{2}})^2+(\gamma_2-\frac{P_1-P_2}{\sqrt{2}})^2]}\|q,\beta\rangle \\ &= \langle q,\beta\|\int \frac{d^2\xi}{\kappa\pi}e^{-\frac{1}{\kappa}[(\gamma_1-\xi_1)^2+(\gamma_2-\xi_2)^2]}|\xi\rangle\langle\xi\|q,\beta\rangle. \end{aligned} \tag{49}$$

Thus using (27) we obtain

$$\begin{aligned} P(\gamma) = \frac{C_q^2}{\kappa\mu^2}&\sum_{m,n=0}^{\infty}\frac{\beta^n\beta^{*m}}{n!\,(q+n)!m!\,(q+m)!} \\ &\times\int\frac{d^2\xi}{\pi}H_{n,q+n}\left(\xi/\mu,\xi^*/\mu\right)H_{q+m,m}\left(\xi/\mu,\xi^*/\mu\right) \\ &\times\exp\left[-|\xi|^2/\mu^2-\frac{1}{\kappa}|\gamma-\xi|^2\right]. \end{aligned} \tag{50}$$

From (47) and (50) we see that two marginal distributions of entangled Husimi function of the state $\|q,\beta\rangle$ are Gaussian-broadened versions of the Wigner marginal distributions obtained based on the entangled state representation [15]. Therefore, for an entangled particle system its marginal distributions of the Husimi function also give the probability of finding the particles in the $\rho-\gamma$ phase space.

In summary, in this paper using the Wigner operator in the entangled state representation and the newly introduced form of the entangled Husimi operator, we have derived Wigner function and Husimi function of the SPCS and its marginal distribution properties. It seem to be a new method and also suitable to get Wigner functions and Husimi functions of other entangled quantum states. The study show that the structure of the Wigner distribution is directly correlated to the quantum interference term of the Wigner function and the squeezing parameter μ. By comparing the two quasibability functions we found that, for the same amount of information included in two functions, the solving process of the Husimi function is simpler than that of the Wigner function.

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